A Lower Bound for Algebraic Connectivity based on Connection Graph Stability Method

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Abstract

In this paper a tight lower bound for algebraic connectivity of graphs (second smallest eigenvalue of the Laplacian matrix of the graph) based on *connection-graph-stability* method is introduced. The connection-graph-stability score for each edge is defined as the sum of the length of all the shortest paths making use of that edge. We prove that the algebraic connectivity of the graph is lower bounded by the size of the graph divided by the maximum connection graph stability of the edges.

Key words: Algebraic connectivity, Graph Laplacian, Connection graph stability score
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1. Introduction

Let $G = (V, E)$ be a connected simple graph with $n = |V|$ vertices and $|E|$ edges. For a graph G a simple path with minimum length connecting two nodes u and v is called shortest path between the nodes and denoted by P_{uv}. The longest shortest path between pairs of nodes is called *diameter* of the graph, denoted by D_{max}. The Laplacian matrix of G is defined as $L = D-A$, where A is the binary adjacency matrix and $D = diag(d_u; u \in V)$ is the degree-diagonal matrix of G. L is a positive semidefinite, symmetric and singular matrix whose eigenvalues are in the form of $\lambda_n(G) \geq \lambda_{n-1}(G) \geq$

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$\ldots \lambda_2(G) \geq \lambda_1(G) = 0$. These eigenvalues are important in graph theory and have close relations to numerous graph invariants. Among them, λ_2, also called algebraic connectivity, has attracted more attention. There were some attempts to find a lower bound for λ_2 based on the simple properties of graphs such as diameter, order, and number of edges of the graph (see [1] for a comprehensive review). For example, Mohar [5] showed that $\lambda_2 \geq \frac{4}{nD_{max}}$ and recently Lu [4] proved that $\lambda_2 \geq \frac{2n}{2+(n-1)nD_{max}-2|E|D_{max}}$. In this paper, we present a new lower bound for λ_2 based on the connection-graph-stability scores associated to the edges, which are defined for each edge as the sum of the length of all the shortest paths making use of that edge. Then, we will prove that the proposed lower bound is always tighter than the previously mentioned bounds proposed by Mohar and Lu.

2. Lower Bound based on Connection-graph-stability method

The connection-graph-stability method was proposed by Belykh et al [6] to establish a criteria for the global stability of synchronization manifold of a network of coupled dynamical systems. Here, we will reuse the concept to obtain a lower bound for algebraic connectivity of a graph.

Definition 1 (Connection-graph-stability). *If for each pair of nodes in the graph a simple connection path is considered (not necessarily the shortest path), a Connection-graph-stability score for each edge k of the graph is denoted by C_k and defined as the sum of the length of the paths passing through the edge, i.e.*

$$C_k = \frac{1}{2}\sum_{u=1}^{n}\sum_{v=1}^{n}\varphi_{uv}(k)|P_{uv}|, \tag{1}$$

where

$$\varphi_{uv}(k) = \begin{cases} 1 & if \quad k \in P_{uv} \\ 0 & otherwise \end{cases}$$

Theorem 1. *Let G be a connected simple graph with n nodes. Then,*

$$\lambda_2 \geq \frac{n}{C_{max}}, \tag{2}$$

where C_{max} is the maximum connection-graph-stability score assigned to the edges of G, i.e. $C_{max} = \max_{k\in E} C_k$.

Proof. Let G be a simple graph. Fiedler [3] showed that

$$\lambda_2 = \min \frac{2n \sum_{uv \in E} (x_u - x_v)^2}{\sum_{u \in V} \sum_{v \in V} (x_u - x_v)^2}, \tag{3}$$

where the minimum is taken over all non-constant vectors $x = (x_v)_{v \in V(G)}$ with $\|x\| = 1$.
Denoting $X_{uv} = x_u - x_v$, equation (3) can be written as

$$\lambda_2 = \min \frac{2n \sum_{uv \in E} (X_{uv})^2}{\sum_{u \in V} \sum_{v \in V} (X_{uv})^2}. \tag{4}$$

Let $P_{uv} = um_1m_2...m_kv$ with $m_i \in V$ being the path connecting u to v and $|P_{uv}|$ its length. X_{uv} can be expressed as

$$X_{uv} = X_{um_1} + X_{m_1m_2} + ... + X_{m_kv} = \sum_{e \in P_{uv}} X_e.$$

Applying the Cauchy-Schwartz inequality, one obtains

$$X_{uv}^2 = (\sum_{e \in P_{uv}} 1.X_e)^2 \leq |P_{uv}| \sum_{e \in P_{uv}} X_e^2.$$

Then,

$$\sum_{u=1}^{n} \sum_{v=1}^{n} (X_{uv})^2 \leq \sum_{u=1}^{n} \sum_{v=1}^{n} (|P_{uv}| \sum_{e \in E} \varphi_{uv}(e) X_e^2) = \sum_{e \in E} 2C_e X_e^2 \leq 2C_{max} \sum_{e \in E} X_e. \tag{5}$$

Substituting (5) in (4)

$$\lambda_2 = \min \frac{2n \sum_{uv \in E} (x_u - x_v)^2}{\sum_{u \in V} \sum_{v \in V} (x_u - x_v)^2} \geq \min \frac{2n \sum_{uv \in E} (x_u - x_v)^2}{2C_{max} \sum_{uv \in E} (x_u - x_v)^2},$$

and finally $\lambda_2 \geq \frac{n}{C_{max}}$.

□

Definition 2 (Alternative Paths). *The alternative paths of any path P_{uv} between two vertices u and v are the paths connecting u and v with length less*

or equal to P_{uv}, e.g. all shortest paths between u and v are alternative paths for each other. The number of all possible shortest paths between u and v is called the number of alternative shortest paths between u and v and denoted by n_{uv}. We denote the set of all possible shortest paths between u and v by $\mathbb{P}_{uv} = \{P_{uv}^{(1)}, P_{uv}^{(2)}, ..., P_{uv}^{(n_{uv})}\}$ (i.e.$n_{uv} = |\mathbb{P}_{uv}|$).

Definition 3 (Path weighting strategy). *For any pair $u, v \in V$ choose a vector $\alpha_{uv} = (\alpha_{uv}^{(1)}, \ldots, \alpha_{uv}^{(n_{uv})})$ such that $\alpha_{uv}^{(q)} \geq 0$ and $\sum_{q=1}^{n_{uv}} \alpha_{uv}^{(q)} = 1$. The corresponding path weighting strategy is denoted by α, the set of all these vectors.*

Definition 4 (Extended connection-graph-stability score). *Then the extended connection graph stability score $C_k(\alpha)$ for edge k is the sum of the weighted lengthes of all shortest paths making use of edge k, i.e.*

$$C_k(\alpha) = \frac{1}{2} \sum_{u=1}^{n} \sum_{v=1}^{n} |P_{uv}| \sum_{q=1}^{n_{uv}} \varphi_{uv}^{(q)}(k) \alpha_{uv}^{(q)}, \tag{6}$$

where

$$\varphi_{uv}^{(q)}(k) = \begin{cases} 1 & if \quad k \in P_{uv}^{(q)} \\ 0 & \quad otherwise \end{cases}$$

and α is the corresponding path weighting strategy.

Theorem 2. *Let G be a connected simple graph with n nodes. Then, for any path weighting strategy α we have*

$$\lambda_2 = a(G) \geq \frac{n}{C_{max}(\alpha)}, \tag{7}$$

where $C_{max}(\alpha)$ is the maximum extended connection-graph-stability score assigned to the edges of G.

Proof. Consider $\mathbb{P}_{uv}$ and n_{uv} for two arbitrary vertices. Let $P_{uv}^{(q)} = um_1m_2...m_kv$ (with $m_i \in V(G)$) be the q_{th} path that connects u to v with length $|P_{uv}^{(q)}| = |P_{uv}|$. Using the q_{th} shortest path, X_{uv}, can be expressed as
$X_{uv} = x_u - x_v = X_{um_1} + X_{m_1m_2} + ... + X_{m_kv} = \sum_{e \in P_{uv}^{(q)}} X_e$.

Using the Cauchy-Schwartz inequality

$$(X_{uv})^2 = (\sum_{e \in P_{uv}^{(q)}} 1.X_e)^2 \leq |P_{uv}| \sum_{e \in P_{uv}^{(q)}} (X_e)^2. \tag{8}$$

On the other hand, one can express X_{uv} as weighted average of its alternative expansions as follows

$$X_{uv}^2 = \sum_{q=1}^{n_{uv}} \alpha_{uv}^{(q)} (X_{uv})^2 \leq \sum_{q=1}^{n_{uv}} \alpha_{uv}^{(q)} |P_{uv}| \sum_{e \in P_{uv}^{(q)}} (X_e)^2, \tag{9}$$

Hence,

$$\sum_{u=1}^{n} \sum_{v=1}^{n} (X_{uv})^2 \leq \sum_{u=1}^{n} \sum_{v=1}^{n} \sum_{q=1}^{n_{uv}} (\alpha_{uv}^{(q)} |P_{uv}| \sum_{e \in P_{uv}^{(q)}} X_e^2)$$

$$= \sum_{u=1}^{n} \sum_{v=1}^{n} |P_{uv}| \sum_{q=1}^{n_{uv}} \sum_{e \in E} (\varphi_{uv}^{(q)}(e) \alpha_{uv}^{(q)} X_e^2) = \sum_{e \in E} 2C_e(\alpha) X_e^2$$

$$\leq 2C_{max}(\alpha) \sum_{e \in E} X_e^2. \tag{10}$$

Substituting (10) in (4), we obtain

$$\lambda_2 = \min \frac{2n \sum_{uv \in E} (X_{uv})^2}{\sum_{u \in V} \sum_{v \in V} (X_{uv})^2} \geq \min \frac{2n \sum_{uv \in E} (X_{uv})^2}{2C_{max}(\alpha) \sum_{uv \in E} (X_{uv})^2},$$

and finally $\lambda_2 \geq \frac{n}{C_{max}(\alpha)}$.
□

Corollary 1. *For any connected simple graph G and any path weighting strategy α we have*

$$\frac{n}{C_{max}(\alpha)} \geq \frac{n}{C_{max}}.$$

Proof. Proof is straight forward since the connection-graph-stability score is a special case of the extended connection-graph-stability score. □

3. Comparing with the other lower bounds

3.1. Mohar's lower bound

Mohar [5] showed that for any connected simple graph of order n and diameter D_{max}

$$\lambda_2 \geq \frac{4}{nD_{max}}. \tag{11}$$

Theorem 3. *For any connected graph G with n vertices, diameter D_{max}, and maximum connection-graph-stability number C_{max}, we have*

$$\frac{n}{C_{max}} \geq \frac{4}{nD_{max}}.$$

Proof. Consider the set of all shortest paths passing through an edge v_1v_2. Define two subsets of vertices as follows: $v \in V_1$ if there is a shortest path from v_2 to v containing v_1 and $v \in V_2$ if there is a shortest path from v_1 to v containing v_2. Note that $v_1 \in V_1$ and $v_2 \in V_2$. Then $V_1 \cap V_2 = \emptyset$.

If this is not the case, there is a vertex $v \in V_1 \cap V_2$. Then there is shortest path $vP_1v_2v_1$ from v to v_1 and a shortest path $vP_2v_1v_2$ from v to v_2. If $|P_1| \leq |P_2|$ then the path vP_1v_2 is a shorter path from v to v_2 than $vP_2v_1v_2$ which is a contradiction. If $|P_2| \leq |P_1|$ we come to the same contradiction.

Now let $m_1 = |V_1|$ and $m_2 = |V2|$, then there are at most m_1m_2 shortest paths between V_1 and V_2 passing through e, hence $C_e \leq m_1m_2D_{max}$, where C_e denotes the connection-graph-stability score of edge e. In addition, $m_2 \leq n-m_1$, thus $C_e \leq m_1(n-m_1)D_{max}$, which is maximized for $m_1 = \frac{n}{2}$, therefore for any edge e of the graph, and hence for the edge that has the maximum connection-graph-stability score, we have

$$C_e \leq (\frac{n}{2})^2 D_{max} \Rightarrow \frac{n}{C_e} \leq \frac{n}{(\frac{n}{2})^2 D_{max}} = \frac{4}{nD_{max}}.$$

□

3.2. Lu's lower bound

Lu [4] obtained the following bound for λ_2 of the Laplacian of any connected simple graph of order n, the number of edges $|E|$ and the diameter D_{max}

$$\lambda_2 \geq \frac{2n}{2 + (n-1)nD_{max} - 2|E|D_{max}}. \quad (12)$$

Theorem 4. *For any connected graph G with n vertices, number of edges $|E|$, diameter D_{max} and maximum connection-graph-stability score C_{max}, we have*

$$\frac{n}{C_{max}} \geq \frac{2n}{2 + (n-1)nD_{max} - 2|E|D_{max}}.$$

Proof. The total number of the shortest paths in G is $\frac{n(n-1)}{2}$ where out of these paths, there are $|E|$ ones with length equal to one. Suppose that e is the edge corresponding to C_{max}. There is only one path of length one passing through this edge (the path that connects adjacent vertices of e). At the same time, at most $\frac{n(n-1)}{2} - |E|$ paths with length more than one can make use of e. According to the definition of the C_{max}, the connection-graph-stability score of e is equal to the sum of the length of these paths plus the length of the path connecting adjacent vertices of e, which is one. Recall the maximum path length, i.e. diameter D_{max}, thus,

$$C_{max} \leq 1 + (\frac{n(n-1)}{2} - |E|)D_{max}$$

$$\Rightarrow \frac{n}{C_{max}} \geq \frac{2n}{2 + (n-1)nD_{max} - 2|E|D_{max}}.$$

□

4. Application to some well-known graphs

The maximum connection-graph-stability score of some well-known graphs can be calculated analytically [6], thus the proposed bound can also be calculated for such graphs. Table 1 summarizes the results on Complete, Path, Cycle, Star, and Peterson graphs.

Table 1: Algebraic connectivity, maximum connection-graph-stability number and proposed lower bound for some well-known graphs

Graph	λ_2	Mohar	Lu	The Lower Bound (1)
Complete graph	n	$\frac{4}{n}$	n	n
Path (n is even)	$2(1-\cos(\frac{\pi}{n}))$	$\frac{4}{n(n-1)}$	$\frac{2n}{2+(n-2)(n-1)^2}$	$\frac{8}{n^2}$
Cycle (n is odd)	$2(1-\cos(\frac{2\pi}{n}))$	$\frac{8}{n(n-1)}$	$\frac{2n}{2+n(n-1)(\frac{n-3}{2})}$	$\frac{24n}{(n^2-1)}$
Star	1	$\frac{2}{n}$	$\frac{n}{1+(n-2)(n-1)}$	$\frac{n}{2n-3}$
Peterson graph	2	0.2	0.164	1.11

5. Conclusion

In this paper a novel lower bound for algebraic connectivity of graphs based on the connection-graph-stability method was presented. It was proved

that the maximum connection-graph-stability score which is defined as the maximum sum of the length of all the shortest paths making use of an edge is inversely related to the algebraic connectivity of the graph. In addition, it is proved that the proposed lower bound is always larger than the bounds proposed by Mohar [5] and Lu [4]. From complexity point of view, the connection-graph-stability score can be interpreted as weighted edge-betweenness-centrality measure were each path is weighted by its length. By this interpretation, the connection-graph-stability scores can be calculated in polynomial time for each edge using slightly modified version of Brandes [2] algorithm which has $O(NE)$ computational complexity.

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